Lecture 8 **2012/11/19**

Outline

1. **Motivation**
2. **Stability of Linear Systems**
3. **Summary**

1. Motivation

- Stability is a global property of systems. For a linear system with constant coefficients, it is solved completely.
- Geometric property is presented: Space decomposition based on stability.

2. Stability of Linear Systems

1) General Definition of Lyapunov Stability

Consider

$$x' = f(t,x), \quad t \geq 0, \quad x \in R^n,$$

where f is continuous and local Lipschitz on x. Suppose that $f(t,0) \equiv 0$, $\forall\, t \geq 0$, i.e. $x = 0$ is an equilibrium point.

Remark 8.1 If $f(t, x_0) \equiv 0$, $\forall\, t \geq 0$, $y = x - x_0$ transforms $x' = f(t,x)$ into the form

$$y' = f(t, y + x_0) \overset{def.}{=} \tilde{f}(t, y) \text{ with } \tilde{f}(t,0) \equiv 0.$$

Therefore, the work on $x = 0$ is also applicable for $x = x_0$. So it is always assumed the origin being equilibrium without loss of generality.

Definition 8.1 The equilibrium $x = 0$ of $x' = f(t,x)$ is said to be

- **stable** if $\forall\, \varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ s.t.

$$\| x(t_0) \| < \delta \;\Rightarrow\; \| x(t, t_0, x_0) \| < \varepsilon, \quad t \geq t_0 \geq 0; \tag{8.1}$$

- **uniformly stable** if $\forall\, \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ s.t. (8.1) is satisfied;
- **asymptotically stable** if $x = 0$ is stable, and there exists $\eta(t_0) > 0$ s. t. for any

x_0 with $\| x_0 \| < \eta(t_0)$, we have

$$\lim_{t\to\infty} x(t, t_0, x_0) = 0;$$

- **uniformly asymptotically stable** if $x = 0$ is uniformly stable, and there exists $\eta > 0$ s. t. for any x_0 with $\| x_0 \| < \eta$, we have

$$\lim_{t\to\infty} x(t, t_0, x_0) = 0, \text{ uniformly in } t_0;$$

that is, there exists $\eta > 0$ s. t. for $\forall\, \varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ s.t.

$$\| x(t_0) \| < \eta \ \Rightarrow\ \| x(t, t_0, x_0) \| < \varepsilon,\ \ t \ge t_0 + T(\varepsilon);$$

- **globally uniformly asymptotically stable** if $x = 0$ is uniformly stable with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$ and for any η s. t. for $\forall\, \varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ s.t.

$$\| x(t_0) \| < \eta \ \Rightarrow\ \| x(t, t_0, x_0) \| < \varepsilon,\ \ t \ge t_0 + T(\varepsilon).$$

Remark 8.2 In control, only "uniform" stability notions are concerned because of robustness required. In Math, we are maybe interested in all of these notions.

Remark 8.3 All of these are in Lyapunov sense. There are other types of stability notions, which will be discussed later in stability theory.

Remark 8.4 Stability notion is in fact the continuous dependence on an initial state x_0 on $[0, \infty)$. It is a global issue. Thus, only the local conditions, like the continuous and Lipschitz condition, can't imply stability!! We need additional conditions for sure of stability.

Remark 8.5 When $f(t, x) = f(x)$, there is no difference between "uniform" and "non-uniform" for stability.

Remark 8.6 The definitions of Lyapunov stability are based on solutions. Therefore, it is qualitative, and difficult to be operated without solving equations.

2) Stability of Linear Systems with Constant Coefficients

Lemma 8.1 $x = 0$ of $x' = Ax$ (A is a real matrix) is stable if and only if e^{At} is bounded for all $t \ge 0$, i.e. there exists $k > 0$ s.t. $\| e^{At} \| \le k$, $\forall\, t \ge 0$. (**Homework**)

Lemma 8.2 $x=0$ of $x'=Ax$ is asymptotically stable if and only if $\lim_{t\to\infty}\|e^{At}\|=0$. (**Homework**)

Lemma 8.3 $x=0$ of $x'=Ax$ is unstable if and only if $\lim_{t\to\infty} e^{At}=\infty$. (**Homework**)

Remark 8.7 Lemma 8.1 to Lemma 8.3 are based on solutions $x(t)=e^{At}x_0$. However, we can find results based on eigenvalues of A by Lemma 8.1-8.3.

Let $w_j=u_j+iv_j$ be generalized eigenvectors of A corresponding to $\lambda_j=\alpha_j+i\beta_j$, where if $\beta_j=0$, $\Rightarrow$ $v_j=0$. Let

$$P=\{u_1,u_2,\cdots,u_k,u_{k+1},v_{k+1},u_{k+2},v_{k+2},\cdots,u_m,v_m\}$$

be a basis of R^n with $n=2m-k$. By Decomposition Theorems (Theorem 7.2-7.4), we have an explicit solution as follows.

$$x(t)=e^{At}x_0=P\,diag(e^{\alpha_j t}\begin{bmatrix}\cos\beta_j t & \sin\beta_j t\\ -\sin\beta_j t & \cos\beta_j t\end{bmatrix})P^{-1}[I_n+Nt+\cdots+\frac{N^{m-1}t^{m-1}}{(m-1)!}]x_0,$$

where $\beta_j=0$ as $j\le k$, and m is not less than the maximum of the algebraic multiplicities of the eigenvalues.

Theorem 8.1 Let A be a real $n\times n$ matrix with (real or complex) eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ repeated according to their (algebraic) multiplicity.

1) $x=0$ is stable if and only if $\operatorname{Re}\lambda\le 0$, when $\operatorname{Re}\lambda=0$, it has the corresponding Jordan block with dimension 1;
2) $x=0$ is asymptotically stable if and only if $\operatorname{Re}\lambda<0$.
3) $x=0$ is unstable if and only if there exists at least one λ_j^0 s.t. $\operatorname{Re}\lambda_j^0>0$ or for those with $\operatorname{Re}\lambda_j^0=0$ it has its Jordan block with dimension more than 1.

Proof. 1) $\operatorname{Re}\lambda\le 0$, when $\operatorname{Re}\lambda=0$, its Jordan blocks have dimension 1 $\Leftrightarrow$ $\|e^{At}\|\le k$ $\Leftrightarrow$ $x=0$ is stable;

2) $\operatorname{Re}\lambda < 0 \iff \lim_{t\to\infty} \| e^{At} \| = 0 \iff x = 0$ is asymptotically stable;

3) There exists at least one λ_j^0 s.t. $\operatorname{Re}\lambda_j^0 > 0$, or for those with $\operatorname{Re}\lambda_j^0 = 0$, it has its Jordan block with dimension more than 1 $\iff \lim_{t\to\infty} e^{At} = \infty \iff x = 0$ is unstable. □

Remark 8.8 Theorem 8.1 can be operated for checking stability because it is determined by eigenvalues of A.

Definition 8.2 Define

$$E^s = \operatorname{Span}\{u_j, v_j : \alpha_j < 0\} = \bigoplus_{\operatorname{Re}\lambda<0} E_\lambda;$$

$$E^u = \operatorname{Span}\{u_j, v_j : \alpha_j > 0\} = \bigoplus_{\operatorname{Re}\lambda>0} E_\lambda;$$

and

$$E^c = \operatorname{Span}\{u_j, v_j : \alpha_j = 0\} = \bigoplus_{\operatorname{Re}\lambda=0} E_\lambda$$

the stable, **unstable and center subspaces**, respectively.

Theorem 8.2 Let A be a real $n\times n$ matrix. Then,

$$R^n = E^s \oplus E^u \oplus E^c.$$

Moreover, E^s, E^u and E^c are invariant under e^{At} respectively. $e^{At} E^x \subset E^x$ for all $t \in R$, where $x = s, u,$ or c.

Proof. By Direct Sum Theorem, we have $R^n = E^s \oplus E^u \oplus E^c$ if A is real. Since any generalized eigenvector subspace is invariant under A, so it is invariant under e^{At}. It is noted that E^s, E^u and E^c are all composed of some direct sum of certain types of generalized eigenvector subspaces, the result is therefore obtained. □

Remark 8.9 The qualitative behavior of solutions of $x' = Ax$:

- If $x_0 \in E^s$, $x(t) = e^{At}x_0 \subset E^s$ for all $t \in R$, and $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to-\infty} \| x(t) \| = \infty$;
- If $x_0 \in E^u$, $x(t) = e^{At}x_0 \subset E^u$ for all $t \in R$, and $\lim_{t\to-\infty} x(t) = 0$, $\lim_{t\to\infty} \| x(t) \| = \infty$;
- If $x_0 \in E^c$, $x(t) = e^{At}x_0 \subset E^c$ for all $t \in R$, and , $x(t)$ either stays bounded or $\lim_{t\to\pm\infty} \| x(t) \| = \infty$.

Definition 8.3 The mapping $\varphi_t = e^{At} : (x_0 \in) R^n \to (x(t) = \varphi_t(x_0) \in) R^n$ is called **the flow** of $x' = Ax$, where the time t is regarded as a parameter.

Remark 8.10 Fixing x_0 as a parameter, then $x(t) = e^{At} x_0$ represents a trajectory; Fixing t as a parameter, then $x(t) = e^{At} x_0$ represents a flow. Different angles to look at the same object.

Definition 8.4 If all eigenvalues of A have nonzero real parts, then the flow $\varphi_t = e^{At} : R^n \to R^n$ is called **a hyperbolic flow** and the corresponding $x' = Ax$ is called **a hyperbolic linear system**.

Definition 8.5 If $R^n = E^s$ (E^u), the origin is called a **sink** (**source**) for $x' = Ax$.

Example 8.1 The matrix

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

has eigenvectors

$$w_1 = u_1 + iv_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ for } \lambda_1 = -2 + i;$$

$$u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ for } \lambda_2 = 3.$$

Then, $E^s = \text{Span}\{u_1, v_1\}$ is the x_1-x_2 plan, in which the origin is a stable focus, the trajectories in the x_1-x_2 plan spiral forward to the origin and $E^u = \text{Span}\{u_2\}$ is the x_3-axis. The trajectories in R^3 are spiral away from the origin and around the x_3-axis. (See the phase portrait) Obviously, $R^3 = E^s \oplus E^u$. The origin is either a sink or a source.

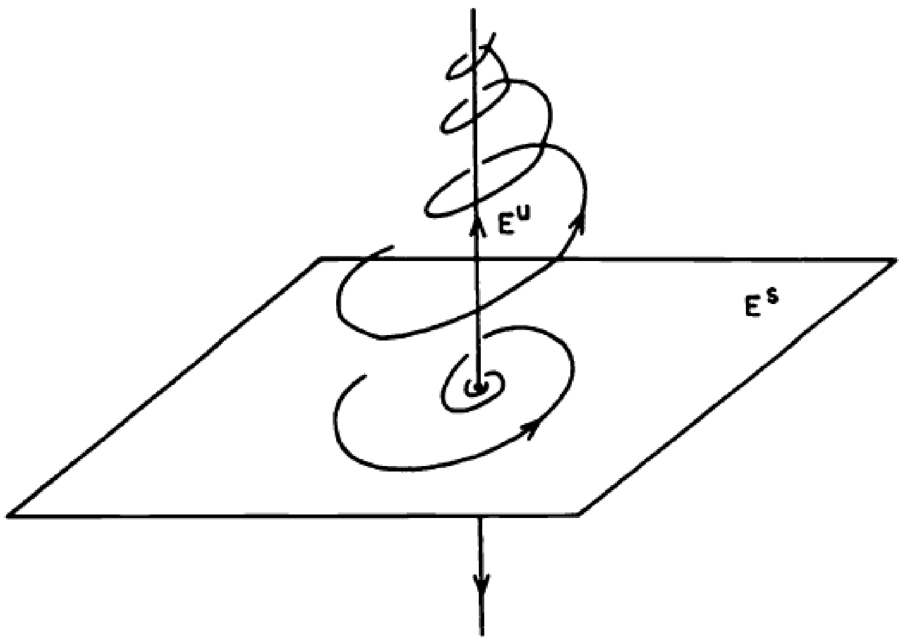

Fig. The stable and unstable subspaces

Example 8.2 The matrix

$$A=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

has eigenvectors

$$w_1 = u_1 + iv_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \quad \lambda_1 = i\,;$$

$$u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda_2 = 2\,.$$

Then, $E^c = \text{Span}\{u_1, v_1\}$ is the x_1-x_2 plan, in which the origin is a center, and $E^u = \text{Span}\{u_2\}$ is the x_3-axis. The trajectories in R^3 are spiral around the cylinders $x_1^2 + x_2^2 = c^2$ away from the origin. (See the phase portrait) Obviously, $R^3 = E^s \oplus E^u$. The origin is either a sink or a source.

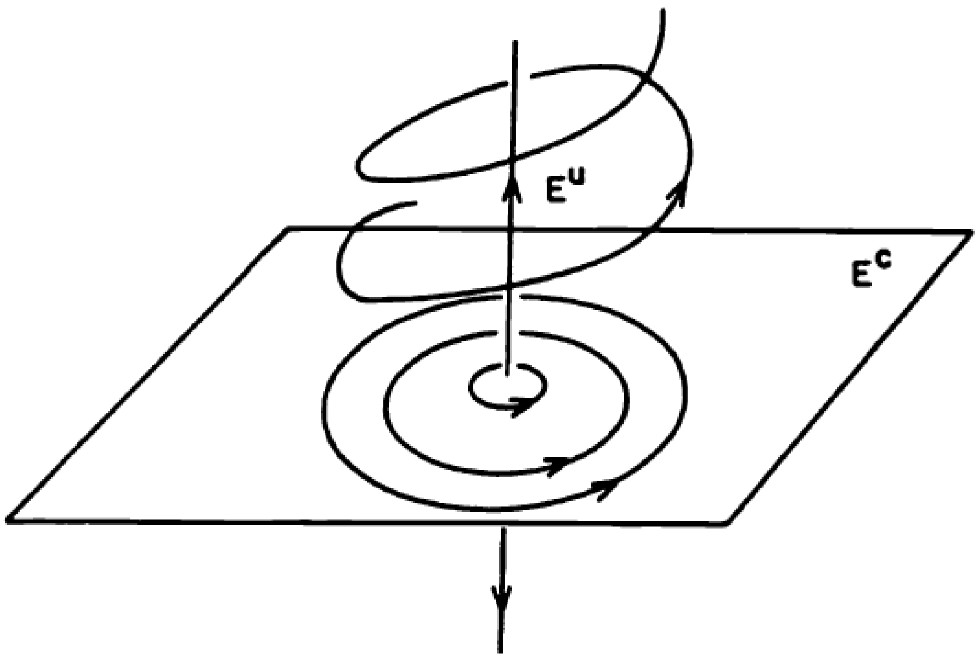

Fig.2 The centre and unstable subspaces

Example 8.3 The matrix

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

has eigenvectors

$$w_1 = u_1 + iv_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{for} \quad \lambda_1 = -2 + i;$$

$$u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{for} \quad \lambda_2 = -3.$$

Then, $E^s = \text{Span}\{u_1, v_1, u_2\} = R^3$. The origin is a sink. (See the phase portrait)

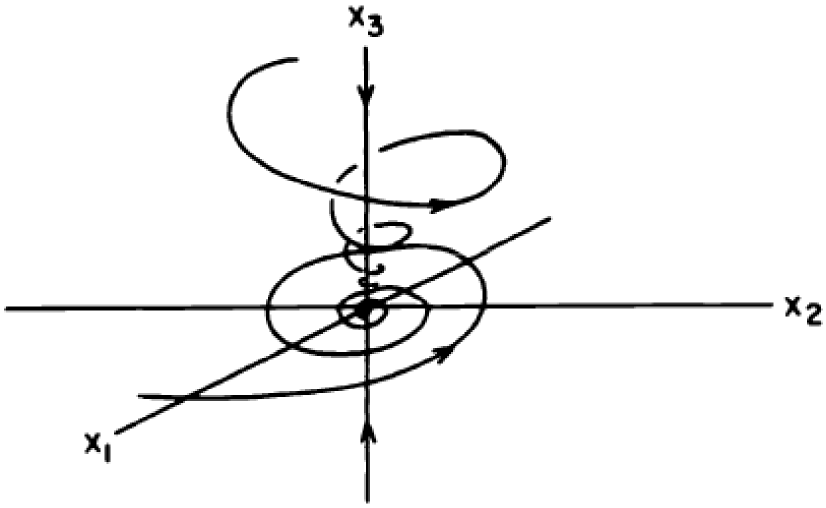

Fig. A linear system with a sink at the origin

3) Stability of Linear Time-Varying Systems

Consider $x' = A(t)x$, where $A(t)$ is continuous on $t \geq 0$.

Theorem 8.3 $x = 0$ is uniformly stable if and only if $\Phi(t, t_0)$ is bounded uniformly for $t_0 \geq 0$, i.e. there exists $k > 0$, independent of $t_0 \geq 0$, s.t.

$$\| \Phi(t, t_0) \| \leq k \quad \text{for} \quad t \geq t_0 \geq 0 .$$

Proof. ($\Leftarrow$) Since $x(t) = \Phi(t, t_0)x_0$, then we have

$$\| x(t) \| = \| \Phi(t, t_0)x_0 \| \leq \| \Phi(t, t_0) \| \| x_0 \| \leq k \| x_0 \| < \varepsilon .$$

From the above, we find $\delta = \dfrac{\varepsilon}{k} > 0$. Then, for $\forall\, \varepsilon > 0$, there exists $\delta = \dfrac{\varepsilon}{k} > 0$ s.t.

$$\| x_0 \| < \delta \;\; \Rightarrow \;\; \| x(t) \| \leq k \| x_0 \| < k\delta = \varepsilon .$$

$x = 0$ is uniformly stable by definition.

($\Rightarrow$) Since $x = 0$ is uniformly stable, then $\forall\, \varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ s.t.

$$\| x_0 \| < \delta \;\; \Rightarrow \;\; \| x(t) \| = \| \Phi(t, t_0)x_0 \| < \varepsilon \quad \text{for} \quad t \geq t_0 \geq 0 .$$

Take $x_{0j} = \dfrac{\delta}{2} e_j$, where $\{e_j\} = \{(0, \cdots, 1, \cdots, 0)^T\}$ is a basis of R^n. Then

$$\| x_{0j} \| = \frac{\delta}{2} < \delta \;\; \Rightarrow \;\; \| \Phi(t, t_0)x_{0j} \| < \varepsilon \quad \text{for} \quad t \geq t_0 \geq 0 .$$

Let $\Phi_j(t, t_0) = \Phi(t, t_0)e_j$. $\Phi_j(t, t_0)$ is a j^{th} column of $\Phi(t, t_0)$. Since

$$\| \Phi_j(t,t_0) \| = \| \Phi(t,t_0) e_j \| = \frac{\delta}{2} \| \Phi(t,t_0) x_{0j} \| < \frac{\delta}{2}\varepsilon ,$$

then, we have

$$\| \Phi(t,t_0) \| \le \sum_{j=1}^{n} \| \Phi_j(t,t_0) \| < n\varepsilon \frac{\delta}{2} = k \quad \text{for} \quad t \ge t_0 \ge 0 ,$$

where k is independent of $t_0 \ge 0$. □

Theorem 8.4 $x=0$ is uniformly asymptotically stable if and only if there exist $k>0$ and $\eta>0$, both independent of $t_0 \ge 0$, such that

$$\| \Phi(t,t_0) \| \le k e^{-\eta(t-t_0)} \quad \text{for} \quad t \ge t_0 \ge 0 .$$

Proof. ($\Leftarrow$) Since $x(t) = \Phi(t,t_0)x_0$, then we have

$$\| x(t) \| = \| \Phi(t,t_0) x_0 \| \le \| \Phi(t,t_0) \| \| x_0 \| \le k \| x_0 \| e^{-\eta(t-t_0)} \quad \text{for} \quad t \ge t_0 \ge 0 .$$

Then, $x=0$ is uniformly asymptotically stable by definition.

($\Rightarrow$) Since $x=0$ is uniformly stable, there exists $k>0$, independent of $t_0 \ge 0$, s.t.

$$\| \Phi(t,t_0) \| \le k , \quad t \ge t_0 \ge 0 ,$$

and $x=0$ is uniformly attractive, i.e. $\lim_{t\to\infty} x(t) = 0$, uniformly for $t_0 \ge 0$ by definition, which implies that $\lim_{t\to\infty} \Phi(t,t_0) = 0$, uniformly for $t_0 \ge 0$. For $\varepsilon = e^{-1} > 0$, there exists $T>0$ s.t.

$$t \ge t_0 + T \quad \Rightarrow \quad \| \Phi(t,t_0) \| < \varepsilon = e^{-1} .$$

This means that $\| \Phi(t,t_0) \| < \varepsilon = e^{-1}$ as long as $t - t_0 \ge T$.

Then, for any $t > t_0$, there exists an integral $\bar{N} > 0$ s.t. $t - t_0 \le \bar{N}T$. Let $N = \min\{ \bar{N} \mid t - t_0 \le \bar{N}T \}$, Then

$$(N-1)T < t - t_0 \le NT .$$

By the property 2) of Theorem 6.8, we have

$$\Phi(t,t_0) = \Phi(t, t_0 + (N-1)T) \Phi(t_0 + (N-1)T, t_0 + (N-2)T) \cdots \Phi(t_0 + T, t_0) .$$

Then, it follows

$$\| \Phi(t,t_0) \| \le \| \Phi(t, t_0 + (N-1)T) \| \| \Phi(t_0 + (N-1)T, t_0 + (N-2)T) \| \cdots \| \Phi(t_0 + T, t_0) \|$$

$$\le k\varepsilon^{N-1} \le eke^{-N} \le eke^{-\frac{t-t_0}{T}} = \bar{k}e^{-\eta(t-t_0)}, \quad t \ge t_0,$$

where $\bar{k} = ke > 0$ and $\eta = \dfrac{1}{T} > 0$ are independent of $t_0 \ge 0$. □

Remark 8.11 Theorem 8.4 shows that, for linear systems, uniformly asymptotic stability is equivalent to its exponential stability.

Remark 8.12 Theorem 8.3 to Theorem 8.4 are also based on solutions $x(t) = \Phi(t, t_0)x_0$. It is conceptually important, not workable for checking stability without solving equations. Can we use eigenvalues like $x' = Ax$ for $x' = A(t)x$. The answer is no in general!

Example 8.4 Counter-example:

$$A(t) = \begin{pmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{pmatrix},$$

It has two eigenvalues: $\lambda_{1,2}(t) = -0.25 \pm i\,0.25\sqrt{7}$ satisfying $\mathrm{Re}\,\lambda_{1,2}(t) \equiv -0.25 < 0$ for all $t \ge 0$. However, its principle matrix solution is solved by

$$\Phi(t,0) = \begin{pmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{pmatrix},$$

and its 2-norm computed by

$$\|\Phi(t,0)\|_2 = \sqrt{\lambda_{\max}(\Phi(t,0)^T\Phi(t,0)},$$

where $\Phi(t,0)^T\Phi(t,0) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}$. So $\|\Phi(t,0)\|_2 = e^{\frac{t}{2}}$ is unbounded. $x = 0$ is unstable by Theorem 8.3.

What about a practical result for $x' = A(t)x$. It is still open so far. However, it has some available results.

Theorem 8.5 If $A(t)$ is symmetric, i.e. $A^T(t) = A(t)$ and continuous on $[0, \infty)$. If the eigenvalues $\lambda_j(t)$ ($j = 1, 2, \cdots, n$) of $A(t)$ satisfy $\max_{1\le j\le n} \lambda_j(t) \le \alpha$, independent of $t_0 \ge 0$ for $t \in [t_0, \infty)$, then we have

$$\|x(t)\|_2 \le e^{\alpha(t-t_0)}\|x(t_0)\|_2, \quad t \ge t_0.$$

In particular, if $\alpha \le 0$, then $x = 0$ is uniformly stable and if $\alpha < 0$, then $x = 0$ is uniformly asymptotically stable.

Proof. Since $A^T(t) = A(t)$, there exists a matrix $Q(t)$ with $Q^T(t) = Q^{-1}(t)$ s.t.

$$Q^T(t)A(t)Q(t) = diag(\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t)),$$

where $\lambda_j(t)$ must be real.

Set $v(t) = Q(t)w(t)$ and we have

$$\left\langle v(t), v(t)\right\rangle = \left\langle Q(t)w(t), Q(t)w(t)\right\rangle = \left\langle w(t), Q^{-1}(t)Q(t)w(t)\right\rangle = \left\langle w(t), w(t)\right\rangle.$$

Then,

$$\left\langle v(t), A(t)v(t)\right\rangle = \left\langle Q(t)w(t), A(t)Q(t)w(t)\right\rangle = \left\langle w(t), Q^{-1}(t)A(t)Q(t)w(t)\right\rangle$$

$$= \left\langle w(t), Q^T(t)A(t)Q(t)w(t)\right\rangle \le \alpha \left\langle w(t), w(t)\right\rangle = \alpha \left\langle v(t), v(t)\right\rangle.$$

Based on $\left\langle v(t), A(t)v(t)\right\rangle \le \alpha \left\langle v(t), v(t)\right\rangle$ and $A^T(t) = A(t)$, we have

$$\frac{d}{dt}\{\| x(t) \|_2^2\} = \frac{d}{dt}\left\langle x(t), x(t)\right\rangle = 2\left\langle x(t), A(t)x(t)\right\rangle \le 2\alpha \left\langle x(t), x(t)\right\rangle$$

$$= 2\alpha \| x(t) \|_2^2.$$

Integrating the inequality on both sides from t_0 to t gives

$$\| x(t) \|_2^2 \le \| x(t_0) \|_2^2 + 2\alpha \int_{t_0}^{t} \| x(s) \|_2^2 \, ds.$$

Therefore, by Gronwall inequality

$$\| x(t) \|_2^2 \le \| x(t_0) \|_2^2 \, e^{2\alpha(t-t_0)}, \quad t \ge t_0.$$

which implies

$$\| x(t) \|_2 \le \| x(t_0) \|_2 \, e^{\alpha(t-t_0)}, \quad t \ge t_0. \quad \square$$

Remark 8.13 The result of Theorem 8.5 looks nice, but conservative. Many symmetric matrices of $A(t)$ don't satisfy the condition of $\max_{1 \le j \le n} \lambda_j(t) \le \alpha$. See examples as follows.

$$A_1(t) = \begin{pmatrix} -1 + 2\cos t & 0 \\ 0 & -1 - 2\cos t \end{pmatrix};$$

$$A_2(t) = \begin{pmatrix} -1 + \max\{2\cos t, -2\cos t\} & 0 \\ 0 & -1 - \max\{2\cos t, -2\cos t\} \end{pmatrix}.$$

$A_1(t)$ and $A_2(t)$ are both symmetric and have the same eigenvalues, satisfying

$$\max\{\lambda_1(t), \lambda_1(t)\} \le \alpha = 1 > 0, \quad \forall\, t \in R .$$

However, the direct computation by solving equations leads $x' = A_1(t)x$ is uniformly asymptotically stable while $x' = A_2(t)x$ is unstable.

Remark 8.14 The examples in Remark 8.13 also illustrate that the stability doesn't only depend on the evolution of the eigenvalues. The evolution of the corresponding (generalized) eigenvectors will also play a crucial role. How to find practical method to test the stability of $x' = A(t)x$ is still open so far!!! See the problem 1 in the Springer book of "Open Problems in Mathematical Systems and Control Theory" by V. D. Blondel, E. D. Sontag, M. Vidyasagar and J. C. Willems, 1998.

Remark 8.15 A useful result for $x' = A(t)x$ is Wazewski inequality:

Let $\lambda_{\max}[A(t)+A^T(t)]$ and $\lambda_{\min}[A(t)+A^T(t)]$ be the maximum and the minimum eigenvalues of $A(t)+A^T(t)$. Then any solution $x(t)$ of $x' = A(t)x$ satisfies

$$\| x_0 \| e^{\frac{1}{2}\int_{t_0}^{t} \lambda_{\min}[A(s)+A^T(s)]ds} \le x(t) \le \| x_0 \| e^{\frac{1}{2}\int_{t_0}^{t} \lambda_{\max}[A(s)+A^T(s)]ds}, \quad t \ge t_0 .$$

Based on this Wazewski inequality, it can develop several results of stability; some of them take Theorem 8.5 as a corollary. However, this inequality sill suffers the same problem mentioned in Remark 8.14.

4. Summary

- For linear systems with constant coefficients, eigenvalues determine stability (in Lyapunov sense).

- $R^n = E^s \oplus E^u \oplus E^c$, each subspace is A-invariant and also flow invariant. Important for geometric analysis of ODE.

- Some important geometric notions: **trajectory** and **flow**; **sink** and **source**; **hyperbolic flow** and **hyperbolic system**; **stable, unstable and center subspace.**

- For linear time varying systems, there is no general method for testing stability. It is still open. However, there exist partial results for stability that are conservative and restrictive. Even for linear periodic systems, it seems no complete solution.

Homework

Prove lemma 8.1-8.3, which is similar to the proof of Theorem 8.3-8.4.